

Loop variables and gravitational Berry's quantum phase in the space-time of a rotating massive body

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Abstract. In this paper we compute the holonomies along curves in the gravitational field of a slowly rotating massive body. We use our results to study the gravitational analogue of Aharonov-Bohm effect in this space-time. We also investigate the behaviour of a scalar quantum particle in this space-time and determine Berry's quantum phase acquired by this particle when transported along a closed curve surrounding the body.

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1. Introduction

The loop space formalism for gauge theories was proposed in the beginning of the seventies [4, 2]. In this formalism the fields depend on curves rather than on space-time points, and a gauge field is described by associating with each curve in space-time an element of the corresponding gauge group. The fundamental quantity that arises in this approach, the non-integrable phase factor [3](loop variable) represents a gauge field more adequately than the field strength or the integral of the vector potential does. In this approach, the electromagnetism, for example, is a gauge-invariant manifestation of the non-integrable phase factor $\exp(\frac{ie}{\hbar c} \oint_c A_\mu dx^\mu)$ (loop variable), which gives the exact description of the theory [3], differently from the field strength and the integral of the potential, which underdescribes and overdescribes, respectively, the theory.

The extension of the loop space formalism to the theory of gravity was firstly considered by Mandelstam[4] who established several equations involving loop variables, and also by Yang[5], Voronov and Makeenko [6] and Bollini et al. [7] who computed the loop variables for the gravitational field corresponding to the Kerr metric. Other investigations on loop variables in the context of gravitational fields includes the computation of this mathematical object in the Schwarzschild-Droste geometry[8], in Taub-NUT space-time[9] and in the background space-time generated by a rotating

black string[10]. There are also studies on the connection between loop variables and gravitomagnetism[11] and clock effects[12].

Loop variables in the theory of gravity are matrices representing parallel transport along curves in a space-time with a given affine connection. They are connected with the holonomy linear transformation which contains important topological informations. These mathematical objects contain information, for example, about how vectors change when parallel transported around a closed curve.

The appearance of topological phases in the quantum dynamics of a single particle moving freely in multiply connected space-times have been studied in a variety of physical systems. The prototype of this phase being the electromagnetic Aharonov-Bohm one[13], which appears as a phase factor in the wave function of an electron which moves around a magnetic flux line. The gravitational analogue of this effect has also been investigated and discussed[14].

In the early eighties Berry discovered[15] that a slowly evolving(adiabatic) quantum system retains informations of its evolution when returns to its original physical state. This information corresponds to what is termed Berry's phase. The appearance of this phase has been generalized to the case of non-adiabatic [16] evolution of a quantum system. In any case the phase depends only on the geometrical nature of the pathway along which the system evolves. This phenomenon has been investigated in several areas of physics[17].

Some works[18] concerning the investigation of Berry's phase in the context of gravitation and cosmology were done in recent years. In particular, Cai and Papini[19] obtained a covariant generalization of the Berry's phase and applied this result to problems involving weak gravitational fields. Recently, Corichi and Pierri[20] studied the behaviour of a quantum scalar particle in a class of stationary space-times and investigated the phase acquired by the particle when transported along a closed path surrounding a rotating cosmic string.

This paper is organized as follows: in Section 2, we calculate the loop variables in the space-time generated by a rotating massive body. In Section 3, we calculate Berry's phase acquired by a scalar quantum particle in this gravitational field. Finally, in Section 4, we present our remarks.

2. Loop variables in the space-time of a rotating massive body

To begin with, let us calculate the loop variables for different curves in the background space-time generated by a rotating massive body. In order to do this we shall consider the following expression for the loop variables[21]

$$U_{AB}(C) = P \exp \left(- \int_B^A \Gamma_\mu(x(\lambda)) \frac{dx^\mu}{d\lambda} d\lambda \right), \quad (1)$$

where Γ_μ is the tetradic connection and A and B are the initial and final points of the curve. Then, associated with every path C from a point A to a point B , we have a loop

variable U_{AB} given by eq.(1) which, by construction, is a function of the curve C as a geometrical object.

As the source of the gravitational field, we consider an infinitely long, infinitely thin massive cylindrical shell which rotates slowly around its axis. The line element corresponding to this situation, in weak field approximation, is given by [22]

$$ds^2 = -(1 - a(r)/2)dt^2 + (1 + a(r)/2)(dr^2 + r^2d\phi^2 + dz^2) + 2b(r)dtd\phi, \quad (2)$$

where $a(r) = -8m\Theta(r - r_0)\ln(r/r_0)$ and $b(r) = 4j \left[\frac{r^2}{r_0^2}\Theta(r_0 - r) + \Theta(r - r_0) \right]$, with $\Theta(x)$ being the unit step function.

The parameter m is the linear mass density and, $j = m\omega r_0$, is the linear angular momentum density, with ω being the angular velocity and r_0 the radius of the cylindrical shell. This approximate solution is justified in a domain with $|a(r)| = |-4\Phi| \ll 1$, where Φ is the Newtonian potential generated by the thin massive cylindrical shell and b the angular momentum of the source, is such that $b^2 \approx 0$.

The Riemann curvature tensor outside the shell is completely determined by the Newtonian potential, in the weak field approximation, if we neglect terms containing products of the form $ba(r)$, $a(r)^2$ and b^2 . This means that in this approximation, the local effects of curvature associated with the rotation of the shell are absent outside it.

Now, let us compute the holonomy for general curves in the xy -plane. To do this let us first compute the tetradic connections. Defining the 1-forms

$$\begin{aligned} \omega^0 &= (1 - a/4)dt + bd\phi, \\ \omega^1 &= (1 + a/4)dr, \\ \omega^2 &= (1 + a/4)r d\phi, \\ \omega^3 &= (1 + a/4)dz \end{aligned} \quad (3)$$

and using the Cartan structure equations, $d\omega^a = -\omega_b^a \wedge \omega^b$, we obtain the tetradic connections which are given by

$$\begin{aligned} \Gamma_{\mu 1}^2 dx^\mu &= [1 - 2m(1 - a/4)] d\phi = -\Gamma_{\mu 2}^1 dx^\mu, \\ \Gamma_{\mu 1}^3 dx^\mu &= -2m/r(1 - a/4)dz = -\Gamma_{\mu 3}^1 dx^\mu, \\ \Gamma_{\mu 1}^0 dx^\mu &= 2m/r(1 - a/4)dt = \Gamma_{\mu 0}^1 dx^\mu, \end{aligned} \quad (4)$$

where $a(r) = -8m \ln(r/r_0)$ and $b(r) = 4m\omega r_0$, outside the shell.

Following the usual procedure to calculate the holonomy[21], we shall first consider circles in the plane perpendicular to the cylinder with center at the origin, with fixed values of t and z . So, in this case we have

$$\Gamma_\mu dx^\mu = \Gamma_\phi d\phi, \quad (5)$$

with Γ_ϕ given by

$$\Gamma_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_\phi & 0 \\ 0 & -A_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

where $A_\phi = [1 - 2m(1 - a/4)]$.

As Γ_ϕ is independent of ϕ , the linear holonomy, U_L , for these circles is

$$\begin{aligned} U_L &= P \exp\left(\int_0^{2\pi} \Gamma_\phi d\phi\right) = \exp(2\pi\Gamma_\phi) \\ &= I + \frac{\Gamma_\phi}{A_\phi} \text{sen}(2\pi A_\phi) + \frac{\Gamma_\phi^2}{A_\phi^2} [1 - \cos(2\pi A_\phi)], \end{aligned} \quad (7)$$

where we used the fact that $(\Gamma_\phi)^3 = -\Gamma_\phi$. Therefore in matrix form the linear holonomy is given by

$$U_L(C) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi A_\phi & \text{sen} 2\pi A_\phi & 0 \\ 0 & -\text{sen} 2\pi A_\phi & \cos 2\pi A_\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp(-2\pi i A_\phi J_{12}), \quad (8)$$

with J_{12} being the generator of the rotations about the z-axis.

Let us compute $U_L(C)$ for a translation in the z -direction with $dt = dr = d\phi = 0$. In this case we have

$$\Gamma_\mu dx^\mu = \Gamma_z dz, \quad (9)$$

where

$$\Gamma_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_z \\ 0 & 0 & 0 & 0 \\ 0 & -A_z & 0 & 0 \end{pmatrix}, \quad (10)$$

with $A_z = 2mr^{-1}(1 - a/4)$. As Γ_z is independent of z , the loop variable corresponding to a segment that goes from z_1 to z_2 , $U(C)$ is given simply by

$$U_{z_1 z_2}(C) = \exp\left(\int_{z_1}^{z_2} \Gamma_z dz\right) = \exp[-i A_z (z_2 - z_1)] J_{13}, \quad (11)$$

where J_{13} is the generator of the rotations about y -axis.

For a translation in the time, we have $\Gamma_\mu dx^\mu = \Gamma_t dt$, with Γ_t given by

$$\Gamma_t = \begin{pmatrix} 0 & A_t & 0 & 0 \\ A_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (12)$$

where $A_t = 2mr^{-1}(1 - a/4)$. Therefore, for translation in the time between t_1 and t_2 , we have the following expression for the loop variable

$$U_{t_1 t_2}(C) = \exp(-i A_t (t_2 - t_1)) J_{14}, \quad (13)$$

where J_{14} is the generator of boosts in the Oz -direction. Note that the linear holonomy does not notice the angular momentum of the source.

Using these previous results given by eqs.(7), (11) and (13), we can write a general expression for the linear holonomy. In the general case, $U(C)$ reads

$$U(C) = P \exp\left(\frac{-i}{2} \int \Gamma_\mu^{ab}(x) J_{ab} dx^\mu\right), \quad (14)$$

where J_{ab} are the generators of the Lorentz group $SO(3,1)$ and Γ_μ^{ab} are the appropriate tetradic connections.

Now, let us compute the translational holonomy which is associated with the jump in time when we go around the cylinder. Before going on, let us notice that the line element given by (2) can be put into the form

$$ds^2 = -[(1 - a(r)/4)dt - bd\phi]^2 + (1 + a(r)/2)(dr^2 + r^2d\phi^2 + dz^2). \quad (15)$$

Proceeding in analogy with the spinning cosmic string case[21], when we go around the cylinder along a circle from point (t, x, y, z) to point (t', x', y', z') , we have the following relations

$$\begin{aligned} t' &= (1 - a/4)t - 2\pi b, \\ x' &= x \cos(2\pi A_\phi) + y \sin(2\pi A_\phi), \\ y' &= -x \sin(2\pi A_\phi) + y \cos(2\pi A_\phi), \\ z' &= z, \end{aligned} \quad (16)$$

where $A_\phi = [1 - 2m(1 - a/4)]$.

The transformations given by eq.(16) can be cast in the form of a homogeneous matrix multiplication by the following device: let M_B^A be a five dimensional matrix with A, B running from 0 to 4. We take M_μ^ν equal to the rotation matrix given eq.(8), $M_0^0 = -a/4$, $M_4^0 = 2\pi b$ and the other elements are equal to zero, so that

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \\ 1 \end{pmatrix} = (-iA_\phi J_{12} + i\frac{a}{4}M + ibM') \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix} \quad (17)$$

where

$$M = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the matrices M and M' are such that $[M, M'] = 0$, $M^3 = -M$ e $M'^3 = -M'$. Therefore, we can express the total holonomy for circle in the plane perpendicular to the cylinder as

$$\begin{aligned} U(C) &= \exp\left(\frac{a\pi}{2}iM + 2\pi ibM'\right) \exp(-2\pi iA_\phi J_{12}) \\ &= \exp\left(-2\pi iA_\phi J_{12} + \frac{a\pi}{2}iM + 2\pi ibM'\right), \end{aligned} \quad (18)$$

where we are considering the five-dimensional representation of the generator of rotations J_{12} .

Then, if one parallel transports a vector along a circle situated in the plane perpendicular to the massive body and around it, after this process it acquires a phase factor given by eq.(18), which depends on the angular momentum. It is worth calling attention to the fact that the angular momentum does not contribute to the curvature tensor in the weak field approximation. Therefore, a particle moving in the region where the Riemann curvature is different from zero and does not depend on the angular momentum exhibits a gravitational effect associated with the angular momentum. Then, we have an example of a non-local phenomena which corresponds to a gravitational analogue of the Aharonov-Bohm effect[14].

3. Gravitational Berry's quantum phase in the space-time of a rotating massive body

Now, let us calculate the Berry's quantum phase acquired by a scalar particle, in the space-time of a rotating cylinder. The behavior of this particle is determined by the covariant Klein-Gordon equation

$$(\square - M^2)\Psi = 0, \quad (19)$$

where $\square = \frac{1}{\sqrt{-g}}\partial_\mu (\partial\sqrt{-g}g^{\mu\nu}\partial_\nu)$, m is the mass of the particle and $\hbar = 1$ and $c = 1$ were chosen.

In order to do the calculation of the Berry's phase, let us write Klein-Gordon equation in the background space-time under consideration, which is given by

$$\{(1+a)\frac{\partial^2}{\partial t^2} - \frac{2b}{r^2}\frac{\partial^2}{\partial\phi\partial t} - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial z^2} + \frac{m^2(2+a)}{2}\}\Psi = 0. \quad (20)$$

Since the background space-time is time independent, we can look for solutions of the form

$$\Psi = \exp(-iE_nt)\psi_n(r, \phi, z), \quad (21)$$

where E_n are the eigenvalues of energy. Substituting eq.(21) into eq.(20) we obtain an equation for $\psi(r, \phi, z)$. Then, we consider the *ansatz* $\psi_n = \exp(i\xi\phi)\Phi_n$, and thus the equation for $\Phi_n(r, \phi, z)$ reads

$$\begin{aligned} &\left\{\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} - \frac{(2+a)}{2}m^2 + \right. \\ &\left. \frac{1}{r^2}(2i\xi - 2biE_n)\frac{\partial}{\partial\phi} + \frac{1}{r^2}(-\xi^2 + 3bE_n\xi) + (1+a)E_n^2\right\}\Phi_n(r, \phi, z) = 0. \end{aligned} \quad (22)$$

Now, let us take $\xi = bE_n$. This will imply that Φ_n obeys the equation

$$\left\{\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} - \frac{(2+a)}{2}m^2 + (1+a)E_n^2\right\}\Phi_n = 0, \quad (23)$$

where we have neglected terms containing b^2 . It is worth calling attention to the fact that this choice for ξ is such that eq.(23) does not contain contribution coming from the angular momentum in the approximation considered. Thus we factorized out the contribution from the angular momentum of the source. In view of this result, we can use the Dirac phase method and write the solution $\psi_n(r, \phi, z)$ as

$$\psi_n(r, \phi, z) = \exp \left(ibE_n \int_{\varphi_0}^{\varphi} d\phi \right) \Phi_n(r, \phi, z), \quad (24)$$

with $\Phi_n(r, \phi, z)$ being the solution of the eq.(22) which comes out from the Klein-Gordon equation in the background gravitational field given by metric (2), with $b = 0$. In this way we find the solution of a more complicated equation from the solution of a simpler one.

Now, let us calculate Berry's phase acquired by a scalar particle in the space-time under consideration. Taking into account that each eigenmode labeled by n acquires a different geometric phase, and therefore we have a degeneracy of the energy eigenvalues, thus we have to follow the procedure done by Mostafazadeh[23] to appropriately treat this problem using the non-Abelian generalization of the Berry's phase, whose connection is given by

$$\mathcal{A}_n^{IJ} = \langle \Phi_n^I(\mathbf{x} - \mathbf{R}) | \nabla_R \Phi_n^J(\mathbf{x} - \mathbf{R}) \rangle \quad (25)$$

where I and J stand for possible degeneracy label. The vector \mathbf{R} localizes the perfectly reflecting box which confines the quantum system, in relation to the center of the cylinder, and \mathbf{x} localizes the particle relative to the center of the box.

The inner product in (25) may be evaluated using the Dirac phase factor as follows:

$$\begin{aligned} & \langle \psi_n^I(x_i - R_i) | \nabla_R \psi_n^J(x_i - R_i) \rangle = \\ & = i \oint_{\Sigma} dS \{ \psi_n^{*I}(x_i - R_i) [ibE_n \psi_n^J(x_i - R_i) + \nabla_R \psi_n^J(x_i - R_i)] \} \\ & = bE_n \delta_{IJ} dR^2 \end{aligned}$$

where $S = r dr d\phi dz$ and R^2 is the polar angle in the box.

Berry's phase can be obtained from the previous result. It is given by

$$\gamma_n(C) = 2\pi bE_n \quad (26)$$

where C is a curve which involves the cylinder.

4. Final remarks

We have shown by explicit calculation, for a metric corresponding to a slowly rotating massive body that the phase shift acquired by a particle, when parallel transported along a given curve C in this background gravitational field is given by an element of the Lorentz group $SO(3, 1)$ times a factor which depends on the angular momentum of the source. From the results concerning the holonomy for curves in the plane perpendicular

to the cylinder, it comes out a phase which depends on the angular momentum of the source and this corresponds to the gravitational analogue[14] of the Aharonov-Bohm effect.

Berry's phase for a scalar particle induced by the gravitational field under consideration also depends on the angular momentum of the source and was obtained by the non-Abelian generalization[23] of this phase, in order to take into account the degeneracy of the energy levels of the system.

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- [1] Mandelstam S., 1962 Ann. Phys. **19**, 1 ; 1968 Phys. Rev. **175**, 1580.
- [2] Wilson K. 1964 Phys. Rev. **D10**, 2445.
- [3] Wu T. and Yang C. N., 1975 Phys. Rev. **D12**, 3843; 1976 Phys. Rev. **D14**, 437.
- [4] Mandelstam S. , 1962 Ann. Phys. **19**, 25; 1968 Phys. Rev. **175**, 1604.
- [5] Yang C. N. , 1974 Phys. Rev. Lett. **33**, 445 .
- [6] Voronov N. A. and Makeenko Y. M., 1982Sov. J. Nucl. Phys., **36**, 444 .
- [7] Bollini C. G., Giambiagi J. J. and Tiomno J. ,1981 Lett. Nuovo. Cim. **31** 1.
- [8] Rothman T., Ellis G. F. R. and Murugan J.,2001 Class. Quantum Grav. **18**, 1217.
- [9] Bini D., Cherubini C. and Jantzen R. T., 2002 Class. Quantum Grav. **19**, 5481.
- [10] de M. Carvalho A. M. , Moraes F. and Furtado C., 2003 Class. Quantum Grav. **20**, 2063.
- [11] Maartens R., Mashhoon B. and Matravers D. R., 2002 Class. Quantum Grav. **19**, 195.
- [12] Bini D., Jantzen R. T. and Mashhoon B., 2002 Class. Quantum Grav. **19**, 17.
- [13] Aharonov Y. and Bohm D., 1959 Phys. Rev. **115**, 485.
- [14] Ford L. H. and Vilenkin A., 1981 J. Phys.. A: Math. Gen. **14**, 2353; Dowker J. S., 1967 Nuovo Cimento **52**, 129; Anandan J., 1994 Phys. Lett. **A195**, 284; Bezerra V. B. , 1987 Phys. Rev. **D35**, 2031.
- [15] Berry M. V. , 1984 Proc. R. Soc. London, **A392**, 45.
- [16] Aharonov Y.and Anandan J., 1987 Phys. Rev. Lett. **58**, 1593.
- [17] Tomita A. and Chiao R. Y., 1986 Phys Rev. Lett. **57**, 937 , Simon R., Kimble H. J. and Sudarshan E. C. G.,1988 Phys Rev. Lett. **61**, 19; Bitter T. and Dubbers D., 1987 Phys. Rev. Lett. **59**, 251; Tycko R., 1987 Phys. Rev. Lett. **58**, 2281; Bird D. M.and Preston A. R., 1988 Phys. Rev. Lett. **61**, 2863.
- [18] Brout R. and Venturi G., 1989 Phys. Rev. **D39**, 2436; Casadio R.and Venturi G., 1995 Class. Quant. Grav. **12**, 1267 (1995); Datta D. P. , 1993 Phys. Rev. **D48**, 5746;*id.* 1995 Gen. Rel. Grav. **27**, 341; 1993 Mod. Phys. Lett. **A8**,191; *ibid.* , **A8** 601; de Assis J. G., Furtado C. and Bezerra V. B. , 2000 Phys. Rev. **D62**, 045003.
- [19] Cai Y. Q. and Papini G. , 1989 Mod. Phys Lett **A4**, 1143 ; 1990 Class. Quant. Grav. **7**, 269.
- [20] Corichi A. and Pierri M. , 1995 Phys. Rev. **D51**, 5870.
- [21] Bezerra V. B., 1990 Ann. Phys. **203**, 392.
- [22] Frolov V. P. , Skarzhinsky V. D. and John R. W., 1987 II Nuovo Cim., **99B**, 67.
- [23] Mostafazadeh A. , 1998 J. Phys. **A31**, 7829.